

A Subsequence with No Exterior Real Roots for Random Littlewood Polynomials

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Abstract

Let

$$f_n(z) = \sum_{k=0}^n \varepsilon_k z^k,$$

where the coefficients are independent and uniformly distributed on $\{-1, 1\}$. We prove that, almost surely, there are infinitely many even n for which every real zero of f_n lies in $[-1, 1]$. Combining this with Do's almost-sure asymptotic

$$\frac{R_n([-1, 1])}{\log n} \longrightarrow \frac{1}{\pi}$$

yields

$$\liminf_{n \rightarrow \infty} \frac{R_n}{\log n} = \frac{1}{\pi} \quad \text{almost surely.}$$

In particular, $R_n/\log n$ cannot converge almost surely to $2/\pi$. The main ingredient is a recurrence argument for a two-dimensional reflected simple random walk, together with a predictable-sampling independence argument for the unused coefficient at each simultaneous record time.

1 Statement of the result

Let $(\varepsilon_k)_{k \geq 0}$ be independent Rademacher random variables, that is,

$$\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}.$$

For

$$f_n(z) = \sum_{k=0}^n \varepsilon_k z^k,$$

write $R_n(I)$ for the number of real zeros of f_n in an interval I , and write $R_n = R_n(\mathbb{R})$. Zeros may be counted either with or without multiplicity; the argument below only uses the set of their locations.

We take as input the following almost-sure asymptotic due to Do:

$$\frac{R_n([-1, 1])}{\log n} \longrightarrow \frac{1}{\pi} \quad \text{almost surely.} \tag{1}$$

Theorem 1. *Almost surely, there are infinitely many even integers n such that f_n has no real zero outside $[-1, 1]$. Consequently, assuming (1),*

$$\liminf_{n \rightarrow \infty} \frac{R_n}{\log n} = \frac{1}{\pi} \quad \text{almost surely.}$$

In particular, $R_n/\log n$ does not converge almost surely to $2/\pi$.

2 An Abel-summation criterion

We first record a deterministic sign criterion.

Lemma 2. *Let $a_0, \dots, a_n \in \mathbb{R}$, and define*

$$A_m = \sum_{j=0}^m a_j, \quad 0 \leq m \leq n.$$

If $A_m \leq 0$ for every $0 \leq m \leq n$ and $A_0 < 0$, then

$$\sum_{j=0}^n a_j y^j < 0 \quad \text{for every } 0 < y < 1.$$

Proof. Abel summation gives

$$\sum_{j=0}^n a_j y^j = (1-y) \sum_{m=0}^{n-1} A_m y^m + A_n y^n. \quad (2)$$

Every term on the right-hand side is nonpositive. Moreover, the $m = 0$ term in the first sum is $(1-y)A_0 < 0$. Hence the entire expression is strictly negative. $\square$

3 Simultaneous coordinatewise record times

Define

$$S_n = \sum_{k=0}^n \varepsilon_k, \quad T_n = \sum_{k=0}^n (-1)^k \varepsilon_k,$$

with $S_{-1} = T_{-1} = 0$, and set

$$W_n = (S_n, T_n), \quad n \geq -1.$$

We use the coordinatewise partial order on $\mathbb{R}^2$.

Lemma 3 (Simultaneous record times). *Almost surely, there are infinitely many even integers n such that*

$$W_n \leq W_\ell \quad \text{coordinatewise for every } -1 \leq \ell < n, \quad (3)$$

and such that $\varepsilon_n = -1$.

Proof. Group the walk into pairs by defining

$$Y_m = W_{2m-1}, \quad m \geq 0,$$

where $Y_0 = W_{-1} = (0, 0)$. Then

$$Y_{m+1} - Y_m = (\varepsilon_{2m} + \varepsilon_{2m+1}, \varepsilon_{2m} - \varepsilon_{2m+1}),$$

which is uniformly distributed on

$$\{(2, 0), (-2, 0), (0, 2), (0, -2)\}.$$

Thus $(Y_m)_{m \geq 0}$ is a planar simple random walk on the even lattice.

For $i = 1, 2$, let

$$M_m^{(i)} = \min_{0 \leq r \leq m} Y_r^{(i)},$$

and define the displacement from the coordinatewise running minimum by

$$D_m = (Y_m^{(1)} - M_m^{(1)}, Y_m^{(2)} - M_m^{(2)}).$$

After dividing by 2, the process $\tilde{D}_m = D_m/2$ is a Markov chain on $\mathbb{N}_0^2$. From an interior vertex it moves to each of the four nearest neighbors with probability $1/4$. On an axis, an attempted step out of the quadrant becomes a holding move; at the origin, the two outward attempts become holding moves. In particular, for every non-loop nearest-neighbor edge $x \sim y$ in $\mathbb{N}_0^2$,

$$P(x, y) = P(y, x) = \frac{1}{4}.$$

Therefore the chain is reversible with respect to counting measure. Its associated electrical network has conductance $1/4$ on every nearest-neighbor edge of the quadrant; self-loops play no role in the effective resistance.

For $r \geq 0$, let Π_r be the set of edges joining the level $\{(i, j) : i + j = r\}$ to the level $\{(i, j) : i + j = r + 1\}$. The cutsets $(\Pi_r)_{r \geq 0}$ are pairwise edge-disjoint, and

$$\sum_{e \in \Pi_r} c(e) = \frac{1}{4} 2(r + 1) = \frac{r + 1}{2}.$$

Hence the Nash–Williams inequality gives

$$R_{\text{eff}}((0, 0) \leftrightarrow \infty) \geq \sum_{r=0}^{\infty} \left(\sum_{e \in \Pi_r} c(e) \right)^{-1} = \sum_{r=0}^{\infty} \frac{2}{r + 1} = \infty.$$

Thus $\tilde{D}$, and therefore D , is recurrent. Since it starts at $(0, 0)$, it returns to $(0, 0)$ infinitely often almost surely.

It remains to select infinitely many of these returns for which the next unused even-indexed sign is -1 . Let

$$\mathcal{F}_k = \sigma(\varepsilon_0, \dots, \varepsilon_k).$$

For $m \geq 0$, the event $\{D_m = (0, 0)\}$ is $\mathcal{F}_{2m-1}$ -measurable, whereas ε_{2m} is independent of $\mathcal{F}_{2m-1}$. Set

$$A_{2m} = \{D_m = (0, 0), \varepsilon_{2m} = -1\}, \quad A_{2m+1} = \emptyset.$$

Then

$$\mathbb{P}(A_{2m} \mid \mathcal{F}_{2m-1}) = \frac{1}{2} \mathbf{1}_{\{D_m = (0, 0)\}}.$$

Because $D_m = (0, 0)$ infinitely often almost surely,

$$\sum_{k \geq 0} \mathbb{P}(A_k \mid \mathcal{F}_{k-1}) = \infty \quad \text{almost surely.}$$

The conditional Borel–Cantelli lemma therefore implies that A_{2m} occurs for infinitely many m almost surely.

Fix such an m and put $n = 2m$. Since $D_m = (0, 0)$, the point Y_m is coordinatewise no larger than every Y_r with $0 \leq r \leq m$. Since $\varepsilon_{2m} = -1$,

$$W_n = W_{2m} = Y_m - (1, 1).$$

We now compare W_n with every earlier point. For $0 \leq r < m$,

$$W_{2r-1} = Y_r, \quad W_{2r} = Y_r + \varepsilon_{2r}(1, 1).$$

Consequently,

$$W_n = Y_m - (1, 1) \leq Y_r = W_{2r-1}$$

and, because $\varepsilon_{2r} \in \{-1, 1\}$,

$$W_n = Y_m - (1, 1) \leq Y_r - (1, 1) \leq Y_r + \varepsilon_{2r}(1, 1) = W_{2r}.$$

Also $W_n \leq Y_0 = W_{-1}$ and $W_n < Y_m = W_{2m-1}$. Thus (3) holds. Since this construction succeeds for infinitely many m , the lemma follows. $\square$

Remark 4. The independence step does not assert that a coefficient chosen at an arbitrary random time is independent of that time. The point is that the event $\{D_m = (0, 0)\}$ is determined by $\varepsilon_0, \dots, \varepsilon_{2m-1}$, while the tested coefficient is the fresh variable ε_{2m} . This predictable separation is exactly what permits the conditional Borel–Cantelli argument.

4 Exclusion of roots outside the unit interval

Proposition 5. *If an even integer n satisfies (3) and $\varepsilon_n = -1$, then f_n has no real zero in $(-\infty, -1) \cup (1, \infty)$.*

Proof. Consider the reciprocal polynomial

$$q_n(y) = y^n f_n(1/y) = \sum_{j=0}^n \varepsilon_{n-j} y^j.$$

For $0 \leq m \leq n$, the m th prefix sum of its coefficients is

$$\sum_{j=0}^m \varepsilon_{n-j} = S_n - S_{n-m-1},$$

where $S_{-1} = 0$. Since $n - m - 1 < n$, the record condition gives $S_n \leq S_{n-m-1}$. Hence every prefix sum is nonpositive, and the first one is $\varepsilon_n = -1$. Lemma 2 therefore yields

$$q_n(y) < 0 \quad (0 < y < 1).$$

Thus q_n has no zero in $(0, 1)$, so f_n has no zero in $(1, \infty)$.

For the negative half-line, write $y = -x$ with $0 < x < 1$. Then

$$q_n(-x) = \sum_{j=0}^n (-1)^j \varepsilon_{n-j} x^j.$$

For $0 \leq m \leq n$, the corresponding prefix sum is

$$\begin{aligned} \sum_{j=0}^m (-1)^j \varepsilon_{n-j} &= (-1)^n \sum_{k=n-m}^n (-1)^k \varepsilon_k \\ &= (-1)^n (T_n - T_{n-m-1}). \end{aligned}$$

Because n is even, this equals $T_n - T_{n-m-1}$, which is nonpositive by (3). Its first value is again $\varepsilon_n = -1$. Lemma 2 gives

$$q_n(-x) < 0 \quad (0 < x < 1).$$

Therefore q_n has no zero in $(-1, 0)$, and f_n has no zero in $(-\infty, -1)$. $\square$

5 Proof of the main theorem

Proof of Theorem 1. By Lemma 3, almost surely there are infinitely many even integers n satisfying the simultaneous record condition and $\varepsilon_n = -1$. Proposition 5 shows that for every such n ,

$$R_n = R_n([-1, 1]). \quad (4)$$

This proves the first assertion.

Now assume (1). Along the infinite subsequence supplied by (4),

$$\frac{R_n}{\log n} = \frac{R_n([-1, 1])}{\log n} \longrightarrow \frac{1}{\pi}.$$

Hence

$$\liminf_{n \rightarrow \infty} \frac{R_n}{\log n} \leq \frac{1}{\pi}.$$

On the other hand, $R_n \geq R_n([-1, 1])$ for every n , so (1) implies

$$\liminf_{n \rightarrow \infty} \frac{R_n}{\log n} \geq \frac{1}{\pi}.$$

Therefore

$$\liminf_{n \rightarrow \infty} \frac{R_n}{\log n} = \frac{1}{\pi} \quad \text{almost surely.}$$

In particular, the sequence cannot converge almost surely to $2/\pi$. □